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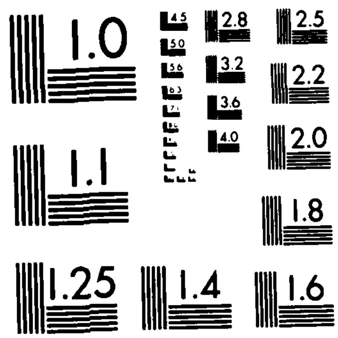

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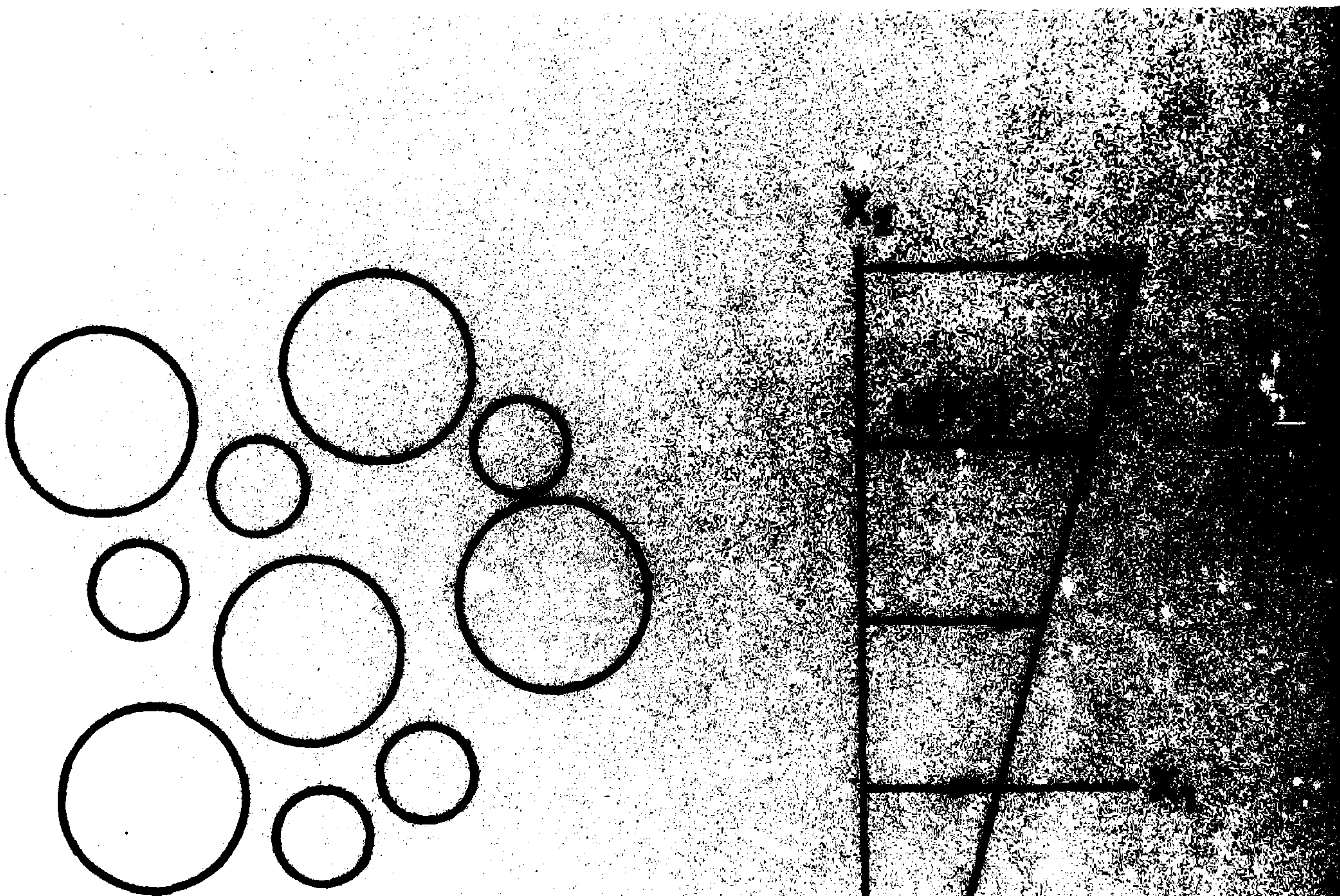
Effect of nonuniform size on internal stresses in a rapid, simple shear flow of granular materials

Part 1. Two grain sizes



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For conversion of SI metric units to U.S./British customary units of measurement consult ASTM Standard E380, Metric Practice Guide, *published by the American Society for Testing and Materials, 1916 Race St., Philadelphia, Pa. 19103.*

Cover: Shear flow of a mixture of two sizes of spheres.

CRREL Report 85-2

February 1985

Effect of nonuniform size on internal stresses in a rapid, simple shear flow of granular materials

Part 1. Two grain sizes

Hayley H. Shen

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Existing theories that predict the stress-strain rate relationship in a rapidly sheared granular flow can only treat materials that are made of single-size particles. However, granular flows usually involve materials of mixed sizes. It has been observed in many laboratory studies that size distribution has a significant effect on the flow of a granular material. Despite its importance, no quantitative theory has been devised that can explain the effect of size distribution. ~~An~~ analytical model is developed ~~here~~ to quantify the stresses in a mixture of spheres with two different sizes and identical material properties. Binary collisions between adjacent particles are considered as the dominating stress-generating mechanism. Comparisons between the theoretical results and the existing laboratory data show good agreement.

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PREFACE

This report was prepared by Dr. Hayley H. Shen, Assistant Professor, Department of Civil and Environmental Engineering, Clarkson University, Potsdam, New York. The research effort was jointly supported by the Engineering Foundation under Grant RI-A-83-02 and the U.S. Army Cold Regions Research and Engineering Laboratory. The latter is where the author spent a year on leave from Clarkson University to work as a Research Physical Scientist in the Snow and Ice Branch.

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NOMENCLATURE

C	total volume concentration
C_L, C_S	volume concentrations of large and small spheres
C_o	densest volume concentration
D_L, D_S	diameters of large and small spheres
$\overline{E}$	average energy dissipation in one sphere
$\overline{E}_{LS}, \overline{E}_{LL}, \overline{E}_{SS}$	average energy dissipation in a pair of spheres
F	total collision frequency
f	collision frequency between surface and exterior spheres
G_L, G_S	number percentages of the large and small spheres
K	constant
$\Delta\overline{M}_{1,2}$	average momentum transfer in the x_1 and x_2 direction
$\Delta\overline{M}_{LS}, \Delta\overline{M}_{SL}$ $\Delta\overline{M}_{LL}, \Delta\overline{M}_{SS}$	average momentum transfer between a pair of like or unlike spheres
$\vec{N}, \vec{P}$	unit vectors in the normal and parallel directions of contact
N	total number density
N_L, N_S	number density of large and small spheres
N_{LS}, N_{LL}, N_{SS}	collision frequency between like or unlike spheres
$N^s_{LS}, N^s_{SL}, N^s_{LL}, N^s_{SS}$	collision frequency between a given sphere and the surrounding spheres
p_2	number density on a surface perpendicular to the x_2-direction
p_L, p_S	number density of large and small spheres on a unit surface
RC	C_S/C_L
RD	D_S/D_L
RG	G_S/G_L
R_L, R_S	average radii of a cell centered at a large or small sphere
s	average gap
u	mean velocity in the x_1-direction
v'_L, v'_S	fluctuation velocity of the large and small spheres
$v_{NL}, v_{PL}, v_{NS}, v_{PS}$	velocity components prior to collision
$v^*_{NL}, v^*_{PL}, v^*_{NS}, v^*_{PS}$	velocity components after collision
x_1, x_2	coordinates
$\alpha, \phi, \psi, \theta$	angles
ϵ	restitution coefficient
μ	kinetic friction coefficient
ρ_s	density of spheres
τ_{21}, τ_{22}	shear and normal stresses
$\tilde{\tau}_{21}$	nondimensional shear stress

EFFECT OF NONUNIFORM SIZE ON INTERNAL STRESSES IN A RAPID, SIMPLE SHEAR FLOW OF GRANULAR MATERIALS
Part 1. Two Grain Sizes

Hayley H. Shen

INTRODUCTION

Many industrial and geophysical processes involve the transport of solids in granular form. Important examples include flows in hoppers, chutes and slurry pipelines, sediment transport in rivers, snow avalanches and debris flows. All of these examples deal with moving particles mixed with air or a liquid, and contain solids that can range in size from boulders to a very fine dust. At present all quantitative descriptions of such flows are largely empirical or semi-empirical (Durand 1953, Newitt et al. 1955, Voellmy 1955, Kennedy and Brooks 1965, Bosley et al. 1969, Roberts 1969).

Recent theories describing the constitutive relationships for rapidly sheared granular mixtures have revealed many insights into the properties of the flows of such mixtures (Jenkins and Satake 1983). These theories have yielded stress-strain rate relationships, and they consider binary collisions between the granular particles to be the major stress-generating mechanism.

Comparisons between these theoretical results and experimental data obtained with various materials (Bagnold 1954, Savage and Sayed 1980, Savage and Sayed 1984) were made by Shen and Ackermann (1982). The experimental data were obtained by shearing dry or neutrally buoyant spherical particles of uniform diameter in a couette device. The comparisons show that current theoretical developments are capable of not only revealing qualitative insights into granular flow, but are also capable of predicting quantitatively measured results quite accurately.

The major limitation to applying the theories to real granular flows is that the constitutive relationships have been developed for granular mixtures of identical size. Almost all real granular flows, however, consist of a wide range of particle sizes. The best an engineer can now do in analyzing a granular flow is to use a somewhat arbitrary "characteristic" size to establish empirical formulas.

Experiments have shown (Gilbert 1914, Durand 1953, Savage and Sayed 1984) that grain size distribution has a significant effect on flow properties. Consider, for example, the data obtained in an experiment in which two mixtures were identical in all respects except one; the first mixture was formed from solids having a grain diameter of 1.32 mm, while the other mixture was formed from two particle sizes (0.6 mm and 1.65 mm) combined to produce a mean particle diameter of 1.34 mm, nearly equal to that of the first mixture (Savage and Sayed 1984). The mixture with a uniform particle size generated stresses that were almost five times higher than those generated with a mixture of two sizes, even though the mean diameters were similar.

In *Part 1* of this study the constitutive equations are developed to describe the stress-strain rate relationships for rapidly sheared mixtures in which the granular solids have two different particle diameters. The extension to a complete distribution of particle sizes will appear in a future paper. In *Part 2* of this study, similar results are obtained for multiple grain size mixtures (Shen 1985). As was done by Jenkins and Savage (1983) and Shen and Ackermann (1982), the binary collisions between particles are considered the major internal momentum transfer mechanism that gives rise to stresses. Comparisons between the theoretical results and experimental data show good agreement.

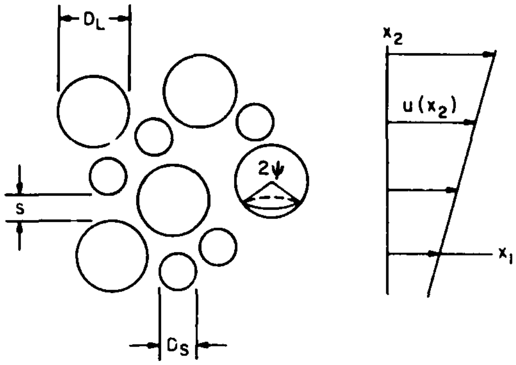

Figure 1. Shear flow of a mixture of spheres.

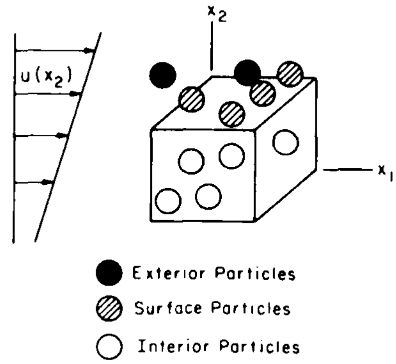

Figure 2. A control volume in a granular flow of uniform spheres.

MODEL OF A RAPIDLY SHEARED GRANULAR FLOW

Consider a simple shear flow of a granular mixture that is steady and uniform as shown in Figure 1. Assume that the solid particles are all made of the same material but have two sizes. Let D_L and D_S represent the diameters of the large and small particles, respectively, and ρ_s equal the density of these particles. Let ϵ be the restitution coefficient and μ the kinetic friction coefficient of the solids. The interstitial fluid effect will be neglected at the moment. The shear flow has a linear velocity profile with $u(x_2)$ representing the mean local velocity at coordinate location x_2. It is also assumed that the body forces are negligible. When the velocity gradient is constant and the body forces negligible, the concentrations of both types of particles can be assumed to be uniform throughout the flow field.

When particles are sufficiently numerous and the shearing rate is sufficiently high, the differences in the mean motion of adjacent particles will trigger collisions among the surrounding particles. These collisions will in turn produce additional collisions and random fluctuations of all other particles through the flow field. A condition is therefore postulated in which every particle has a mean velocity $u(x_2)$ and a continually fluctuating velocity component superimposed on the mean motion. This type of flow field is consistent with flow conditions of a granular mixture postulated in previous analyses (Kanatani 1979, Ogawa et al. 1980, Shen and Ackermann 1982, Jenkins and Savage 1983) and has been observed in experimental studies.* The collisions create a momentum transfer between particles. The rate of the net momentum transfer across any unit surface area can then be interpreted as creating both shear and normal stresses on that area.

As the frequency of collisions rises, the resulting momentum transfer eventually dominates any other mechanism that might also generate stresses. When stresses are generated predominantly by momentum transfer from collisions, the granular flow is described as being in the "grain inertia" region (Bagnold 1954). As in previous analyses (Kanatani 1979, Ogawa et al. 1980, Shen and Ackermann 1982, Jenkins and Savage 1983) the basic assumptions are: binary collisions are the dominating mechanism for stress generation, viscous stresses generated by fluid-fluid shearing are negligible, and the mass transfer and associated momentum transfer are negligible across any surfaces parallel to the flow direction. For mixtures of granular solids having water or air as the interstitial fluid, the concentrations that will make the above assumptions valid should be more than 20% by volume and the shearing rate more than approximately 10 s^{-1}. These criteria have been obtained from the experimental work done by Bagnold (1954) and Savage and Sayed (1980).

The stresses on a unit surface are defined as the total rate of momentum transfer across a unit surface. Under the above assumptions, this momentum transfer is caused by the binary collisions between particles located on the unit surface and particles coming from outside a unit volume

*Personal communication with N.L. Ackermann, Clarkson University, Potsdam, N.Y., 1984.

bounded by that unit surface. For illustration, consider a granular flow of single-size particles. Let p_2 be the number of particles on a unit surface perpendicular to the x_2-direction, as shown by the top surface in Figure 2. Let f be the frequency of binary collisions between a particle on that surface and the particles exterior to that surface. Let $\Delta\overline{M}_1$ be the average momentum transferred in the x_1 direction to the surface particles after each collision. The shear stress on that surface τ_{21} can therefore be computed as

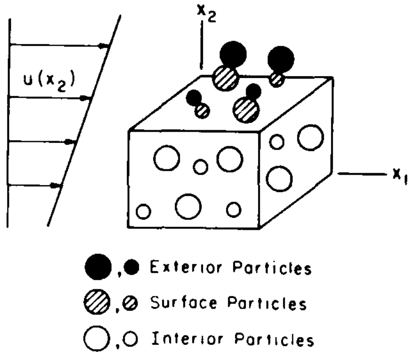

Figure 3. A control volume in a granular flow of mixed spheres.

$$\tau_{21} = p_2 \cdot f \cdot \Delta\overline{M}_1 \ . \tag{1}$$

The normal stress on the surface τ_{22} is computed similarly, with $\Delta\overline{M}_1$ being replaced by $\Delta\overline{M}_2$, where $\Delta\overline{M}_2$ is the average momentum transferred in the x_2-direction. The above equation was first established by Bagnold (1954). The idea used to develop eq 1 can be extended to flows with particles that are not uniform, with the right side of eq 1 being appropriately generalized to incorporate combinations of different particles.

FORMULATION OF STRESSES IN A TWO-SIZE MIXTURE

Consider a unit volume in the shear flow, as shown in Figure 3. On the top surface there are four different kinds of collisions: a large exterior particle colliding with a large surface particle, a large exterior particle colliding with a small surface particle, a small exterior particle colliding with a large surface particle, and a small exterior particle colliding with a small surface particle. Any collision between the surface and exterior particles results in a momentum transfer across the surface. This momentum transfer is attributable to the differences in the initial momentums of the particles. The initial relative momentum of colliding particles depends on not only the difference between their mean velocities, but also the difference in their fluctuating velocities.

The following notations will be used in the ensuing discussion:

i = 1 or 2, indicating the x_1- or x_2-direction.

$\Delta\overline{M}_{LSi}$ = average momentum $\Delta\overline{M}$ transferred in the x_i-direction from the collisions between a large surface particle and a small exterior particle.

$\Delta\overline{M}_{SLi}$ = average momentum transferred in the x_i-direction from the collisions between a small surface particle and a large exterior particle.

$\Delta\overline{M}_{LLi}$ and $\Delta\overline{M}_{SSi}$ = average momentum transferred in the x_i-direction from collisions between a surface particle and an exterior particle of the same kind.

N^s_{LS} = frequency of collisions between a single large surface particle and any of the surrounding small particles.

N^s_{SL} = frequency of collisions between a single small surface particle and any of the surrounding large particles.

N^s_{LL} and N^s_{SS} = frequencies of collisions between a single surface particle and the surrounding particles that are of the same kind.

p_L, p_S = number of large or small particles on a unit surface.

If we assume that the concentration and random motion of both types of particles are uniform, the collision rate between a single surface particle and the exterior particles should be half of the collision rate between a surface particle and all the surrounding particles. Hence, $N^s_{LS}/2$, $N^s_{SL}/2$, $N^s_{LL}/2$ and $N^s_{SS}/2$ indicate the four collision rates of the four different types of collisions between a surface particle and the exterior particles.

Using the notation defined above, we can formulate the stresses as

$$\tau_{21} = \left(\Delta\overline{M}_{LL1}\frac{N^s_{LL}}{2} + \Delta\overline{M}_{LS1}\frac{N^s_{LS}}{2}\right)\rho_L + \left(\Delta\overline{M}_{SL1}\frac{N^s_{SL}}{2} + \Delta\overline{M}_{SS1}\frac{N^s_{SS}}{2}\right)\rho_S \tag{2}$$

$$\tau_{22} = \left(\Delta\overline{M}_{LL2}\frac{N^s_{LL}}{2} + \Delta\overline{M}_{LS2}\frac{N^s_{LS}}{2}\right)\rho_L + \left(\Delta\overline{M}_{SL2}\frac{N^s_{SL}}{2} + \Delta\overline{M}_{SS2}\frac{N^s_{SS}}{2}\right)\rho_S. \tag{3}$$

To determine the collision frequency, the energy balance equation will be utilized. First, a concept of internal energy must be established. The collisions that take place in a rapidly sheared granular flow cause the particles to move randomly. This random motion can be represented by velocity fluctuations about the mean velocity. For the case of uniform particles, let v' be the fluctuation velocity about the mean velocity for any particle. The internal energy per unit volume contained in the solids is then defined as $\frac{1}{2} C \rho_s v'^2$, where C is the volume concentration of solids. In a steady, simple shear flow with a uniform shearing rate, the energy balance is written as (Shen and Ackermann 1982)

$$\tau_{21}\frac{du}{dx_2} = N \cdot F \cdot \overline{E}. \tag{4}$$

On the left side of eq 4 is the rate of work spent on increasing the internal energy of a unit volume in the flow. On the right side of eq 4 is the rate of internal energy dissipation in a unit volume because of collisions. In eq 4 N denotes the number of particles in a unit volume, F the collision frequency per particle and $\overline{E}$ the average energy dissipation per particle per collision.

For a mixture of two different sizes, the energy equation described by eq 4 should be generalized as follows:

$$\tau_{21}\frac{du}{dx_2} = N_{LS}\,\overline{E}_{LS} + N_{LL}\,\overline{E}_{LL} + N_{SS}\,\overline{E}_{SS} \tag{5}$$

where N_{LS} is the rate of collisions between all pairs of large and small particles in a unit volume, while N_{LL} and N_{SS} are the rates of binary collisions between like particles in a unit volume, and $\overline{E}_{LS}$ is the average energy loss in a pair of colliding large and small particles, while $\overline{E}_{LL}$ and $\overline{E}_{SS}$ are the average energy losses in a pair of colliding like particles. After substituting eq 2 into eq 5, we obtain

$$\left[\left(\Delta\overline{M}_{LL1}\frac{N^s_{LL}}{2} + \Delta\overline{M}_{LS1}\frac{N^s_{LS}}{2}\right)\rho_L + \left(\Delta\overline{M}_{SL1}\frac{N^s_{SL}}{2} + \Delta\overline{M}_{SS1}\frac{N^s_{SS}}{2}\right)\rho_S\right]\frac{du}{dx_2}$$

$$= N_{LL}\,\overline{E}_{LL} + N_{LS}\,\overline{E}_{LS} + N_{SS}\,\overline{E}_{SS}\,. \tag{6}$$

The parameters that appear in the above equation are functions of the flow condition and material

properties. These functions can be summarized as follows in eq 7-11:

$$p_{\mathrm{L\ or\ S}} = p_{\mathrm{L\ or\ S}}\,(C_{\mathrm{L}}, C_{\mathrm{S}}, D_{\mathrm{L}}, D_{\mathrm{S}}) \tag{7}$$

where C_L and C_S are the concentrations of the large and small particles and D_L and D_S are their diameters;

$$N^{s}_{\mathrm{PQ}} = N^{s}_{\mathrm{PQ}}\,(C_{\mathrm{L}}, C_{\mathrm{S}}, D_{\mathrm{L}}, D_{\mathrm{S}}, v'_{\mathrm{L}}, v'_{\mathrm{S}})\ ;\ P \text{ and } Q = L \text{ or } S \tag{8}$$

where v'_L and v'_S are the fluctuation velocity of large and small particles, respectively;

$$N_{\mathrm{PQ}} = N_{\mathrm{PQ}}\,(C_{\mathrm{L}}, C_{\mathrm{S}}, D_{\mathrm{L}}, D_{\mathrm{S}}, v'_{\mathrm{L}}, v'_{\mathrm{S}})\ ;\ P \text{ and } Q = L \text{ or } S \tag{9}$$

$$\Delta\overline{M}_{\mathrm{PQi}} = \Delta\overline{M}_{\mathrm{PQi}}\,(\frac{du}{dx_2}, v'_{\mathrm{L}}, v'_{\mathrm{S}}, D_{\mathrm{L}}, D_{\mathrm{S}}, C_{\mathrm{L}}, C_{\mathrm{S}}, \epsilon, \mu, \rho_s)\ ;\ P \text{ and } Q = L \text{ or } S,\ i = 1 \text{ or } 2 \tag{10}$$

and

$$\overline{E}_{\mathrm{PQ}} = \overline{E}_{\mathrm{PQ}}(\frac{du}{dx_2}, v'_{\mathrm{L}}, v'_{\mathrm{S}}, D_{\mathrm{L}}, D_{\mathrm{S}}, \epsilon, \mu, \rho_s)\ ;\ P \text{ and } Q = L \text{ or } S. \tag{11}$$

In eq 7-11 the parameters du/dx_2, D_L, D_S, C_L, C_S, ρ_s, ϵ and μ are specified. The unknown variables are the fluctuation velocities v'_L and v'_S.

After the functional relationships described in eq 7-11 are established, the fluctuation speeds v'_L and v'_S can be determined using eq 6 and by assuming an equipartition of energy. The equipartition of energy is described as

$$\rho_s\ \frac{\pi}{6} D_{\mathrm{S}}^3\ v'^2_{\mathrm{S}} = \rho_s\ \frac{\pi}{6} D_{\mathrm{L}}^3\ v'^2_{\mathrm{L}} \tag{12}$$

which says that the internal energy (the "random fluctuation energy") of the small and large particles is the same. As proved in many texts on the kinetic theory of gases (Kennard 1938, Chapman and Cowling 1970) the random fluctuations of different particles that form an equilibrium mixture contain the same amount of fluctuation energy. In a steady, constant velocity gradient, rectilinear granular flow, the mechanism that generates the random fluctuation energy, is uniform everywhere. Hence, a random energy equilibrium state can be assumed.

The following section is devoted to explicitly quantifying eq 7-11.

MICROSCOPIC KINEMATICS AND DYNAMICS

Number density of large and small spheres on a unit surface–p_L, p_S

Consider a unit surface area perpendicular to the x_2-direction. This unit surface cuts through numerous large and small particles at various depths in the particles. If we assume a completely random situation, the average area cut by the unit surface can be obtained by

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Bibliography: p. 13.
1. Flow. 2. Granular flow. 3. Slurries. I. United States. Army. Corps of Engineers. II. Cold Regions Research and Engineering Laboratory, Hanover, N.H. III. Series: CRREL Report 85-2.

Substituting eq B1-B7 into eq B5 yields

$$\overline{E}_{\mathrm{LS}} = \rho_{\mathrm{s}} \frac{\pi D_{\mathrm{L}}^3}{6} \frac{v_{\mathrm{L}}'^2 + v_{\mathrm{S}}'^2}{1+(\frac{D_{\mathrm{L}}}{D_{\mathrm{S}}})^3} \left[\frac{1-\epsilon^2}{4} + \frac{\mu(1+\epsilon)}{\pi} - \frac{\mu^2(1+\epsilon)^2}{4}\right] \ . \tag{B8}$$

APPENDIX B: AVERAGE ENERGY LOSS IN A PAIR OF COLLIDING LARGE AND SMALL SPHERES

Consider a pair of large and small spheres about to collide, as shown in Figure B1. The total energy loss because of a collision is independent of the inertial reference frame; hence a reference frame is chosen to move with the center of mass of these two spheres. If we assume that only fluctuation components just prior to collision are important in this computation, the velocities of the large and small spheres in the normal and parallel directions of contact are related by (Greenwood 1965)

$$\nu'_{NS} = -\frac{m_L}{m_S}\nu'_{NL} \quad , \quad \nu'_{PS} = -\frac{m_L}{m_S}\nu'_{PL} \ . \tag{B1}$$

After the collision, let * denote the post-collision conditions; the velocities then become (Greenwood 1965)

$$\nu'^{*}_{NS} = -\epsilon\nu'_{NS} \quad , \quad \nu'^{*}_{NL} = -\epsilon\nu'_{NL} \ . \tag{B2}$$

Hence, in the parallel direction the velocities for the large and small spheres become

$$\nu'^{*}_{PL} - \nu'_{PL} = \mu(\nu'^{*}_{NL} - \nu'_{NL}) \tag{B3}$$

and

$$\nu'^{*}_{PS} - \nu'_{PS} = \mu(\nu'^{*}_{NS} - \nu'_{NS}) \ . \tag{B4}$$

The average energy loss in a pair of colliding large and small spheres is

$$\bar{E}_{LS} = \frac{2}{\pi}\int_0^{\pi/2} \frac{m_S}{2}\left[(\nu'^2_{NS}+\nu'^2_{PS}) - (\nu'^{*2}_{NS}+\nu'^{*2}_{PS})\right] + \frac{m_L}{2}\left[(\nu'^2_{NL}+\nu'^2_{PL}) - (\nu'^{*2}_{NL}+\nu'^{*2}_{PL})\right] d\alpha \tag{B5}$$

where

$$\nu'_{NS} = \nu'_S \cos\alpha \quad , \quad \nu'_{NL} = \nu'_L \cos\alpha \tag{B6}$$

and

$$\nu'_{PS} = \nu'_S \sin\alpha \quad , \quad \nu'_{PL} = \nu'_L \sin\alpha \ . \tag{B7}$$

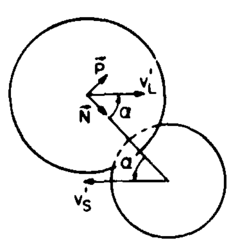

Figure B1. A collision between large and small spheres, center of mass system.

The average momentums transferred from a small sphere to a large sphere after the collision are

$$\Delta\bar{M}_{LS1} = \frac{2}{\pi^2} \int_0^{\pi/2} d\phi \int_{\pi/2}^{3\pi/2} \rho_S \frac{\pi D_L^3}{6} v_{NL}^* (\vec{N} + \mu\vec{P}) \cdot \vec{i}\, d\theta \tag{A6}$$

and

$$\Delta\bar{M}_{LS2} = \frac{2}{\pi^2} \int_0^{\pi/2} d\phi \int_{\pi/2}^{3\pi/2} \rho_S \frac{\pi D_L^3}{6} v_{NL}^* (\vec{N} + \mu\vec{P}) \cdot \vec{j}\, d\theta \,. \tag{A7}$$

The term $\mu\vec{P}$ in eq A7 will disappear under the assumption that $v'_{L \text{ or } S} >> (du/dx_2)[(D_L+D_S)/2]$. This argument was also adopted by Shen and Ackermann (1982). Substituting eq A1-A5 into eq A6 and substituting eq A1-A5 into eq A7, we obtain

$$\Delta\bar{M}_{LS1} = \rho_s \frac{\pi D_L^3}{6} (1+\epsilon)(0.053 + 0.081\,\mu) \frac{du}{dx_2} (\frac{D_L+D_S}{2} + s) \frac{2D_S^3}{D_L^3+D_S^3} \tag{A8}$$

and

$$\Delta\bar{M}_{LS2} = -\rho_s \frac{\pi D_L^3}{6} \frac{4}{\pi^2} \frac{(1+\epsilon)\, D_S^3}{D_L^3 + D_S^3} \sqrt{v_S'^2 + v_L'^2} \,. \tag{A9}$$

In a collision process only impulsive forces are involved; hence the momentum transferred from a large sphere to a small sphere equals that from a small sphere to a large sphere, or

$$\Delta\bar{M}_{LS1 \text{ or } 2} = \Delta\bar{M}_{SL1 \text{ or } 2} \,. \tag{A10}$$

APPENDIX A: AVERAGE MOMENTUM TRANSFER BETWEEN A PAIR OF COLLIDING LARGE AND SMALL SPHERES

Consider a pair of large and small spheres about to collide, as shown in Figure A1. The reference frame is fixed to move with the center of the large sphere at a velocity equal to the velocity of the large sphere just prior to collision. For the large sphere in this reference frame, the post-collision velocities, as indicated by *, in the directions normal and parallel to the contact surface, are (Greenwood 1965)

$$\nu_{NL}^* = \frac{(1+\epsilon)D_S^3}{D_L^3 + D_S^3}\nu_{NS} \quad , \quad \nu_{PL}^* = \mu\frac{(1+\epsilon)D_S^3}{D_L^3 + D_S^3}\nu_{NS} \tag{A1}$$

where ν_{NS} is the normal component of the small particle's velocity. An assumption is made in deriving ν_{PL}^* such that the impulsive frictional force is proportional to the impulsive normal force, with proportionality μ being the kinetic frictional coefficient.

If we follow the argument used in deriving stresses for a mixture of uniform-sized spheres (Shen and Ackermann 1982) when computing the average momentum transfer in the x_1-direction, the velocity of the small particle is assumed to be

$$\vec{\nu}_S = (\frac{D_S+D_L}{2} + s)\frac{du}{dx_2}\cos\phi\,\vec{i} \ . \tag{A2}$$

When computing the average momentum transfer in the x_2-direction, it is assumed that

$$\vec{\nu}_S = \sqrt{\nu_S'^2 + \nu_L'^2}\,(\cos\alpha\vec{N} + \sin\alpha\vec{P}) \quad ; \quad 0 \leqslant \alpha \leqslant \frac{\pi}{2} \ . \tag{A3}$$

The two unit vectors $\vec{N}$ and $\vec{P}$ indicate the normal and parallel directions of collisions. They are quantified as

$$\vec{N} = -\sin\phi(\cos\theta\,\vec{i} - \sin\theta\,\vec{k}) - \cos\phi\,\vec{j} \tag{A4}$$

and

$$\vec{P} = \sqrt{\cos^2\phi + \sin^2\phi\sin^2\theta}\ \ [\vec{i} - \frac{\sin\phi\cos\phi\cos\theta\,\vec{j}}{\cos^2\phi+\sin^2\phi\sin^2\theta} + \frac{\sin^2\phi\cos\theta\sin\theta\vec{k}}{\cos^2\phi+\sin^2\phi\sin^2\theta}] \ . \tag{A5}$$

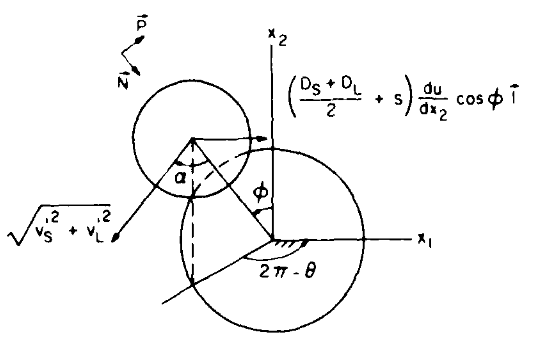

Figure A1. A collision between large and small spheres.

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be included at a crude level as was done by Shen and Ackermann (1982), in which the energy dissipation attributable to the fluid drag was the only mechanism provided by the interstitial fluid.

The results of this work show quantitatively the significance of size gradations. For constant volume concentrations, as the smaller particles decrease to half the original size, the internal shear stress could be reduced by as much as ten times. Although theory checks well with the limited existing data, the general validity of the present theoretical work has to be confirmed by future experimental work for a wider variety of size gradations.

There are a few difficulties, however, regarding the description of particle configurations that may introduce errors in the present analysis. These difficulties are discussed below.

The densest volume concentration C_o that appeared when defining the average gap between particles, and also in the final form of the stresses, is not a constant. Nevertheless, in all the computations it is assumed that C_o is equal to 0.74, which is the densest volume concentration for a hexagonal packing of uniform spheres. This parameter, C_o, should depend on the packing pattern and the size ratio of the two different particles. When the small particles are infinitely small, they can fill the voids between the large spheres. Hence, in the limiting case the densest volume concentration for a two-size packing should be $0.74 + (1-0.74) \times 0.74 = 0.93$. Fortunately, values of the nondimensional stresses are quite insensitive to the variation of C_o. It is found that as $C = 0.5$, when C_o varies from 0.74 to 0.92, the nondimensional shear stress defined in eq 41 increases by less than 25%.

Another difficulty arises when the ratio D_S/D_L tends to be very small, since as the smaller particles become extremely small, a large number of small particles will be surrounded by small particles only. Namely, as D_S/D_L goes to zero, the statistical concept used in estimating the average gap between particles may break down. It is believed that as D_S/D_L goes to zero, the average gap between small particles is overestimated by eq 21. Thus, the actual stresses may be somewhat greater than the theoretically predicted values as D_S/D_L goes to zero.

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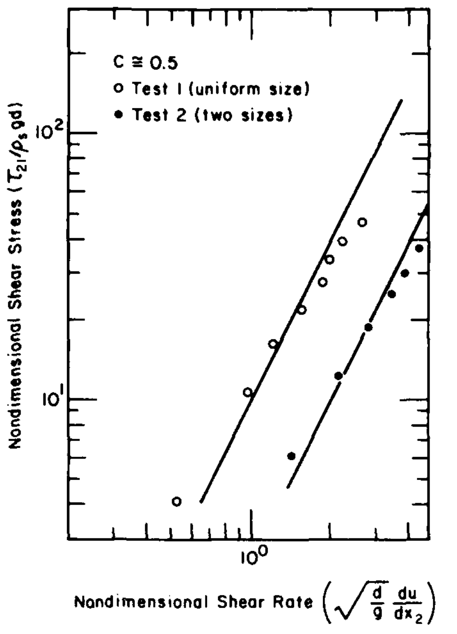

Figure 6. Nondimensional shear stress versus nondimensional shear rate; d = *diameter. (After Savage and Sayed [1984], data from their Figures 9b and 18b.)*

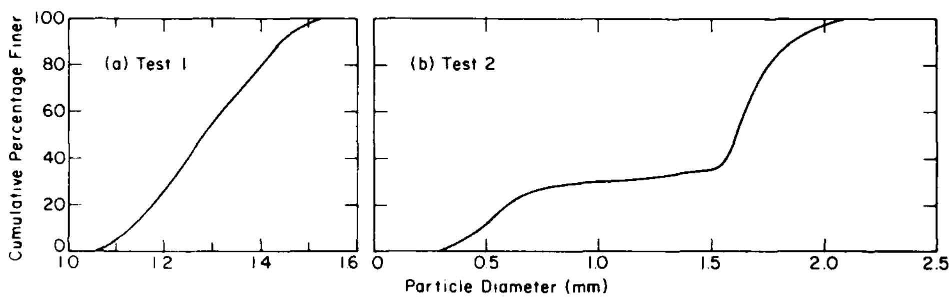

Figure 7. Particle size distribution of spheres (after Savage and Sayed 1984).

This prediction is of the same order of magnitude as the measured data. The discrepancy between the numerical values of the data and the prediction may be the result of the nonuniform size distribution of both tests 1 and 2. As shown in Figure 7a, the actual size of spheres used in test 1 is not uniform. In fact, the spectrum is quite wide. The two-size mixture used in test 2 as shown in Figure 7b is not composed of two distinct sizes either. However, the spectra are narrower than the one in test 1. Based on the present theoretical study, it is expected that if a true uniform-size material is used in test 1, measured stresses would be greater than the values given in Figure 6. Therefore, the ratio given in eq 42 could be closer to that given in eq 43 than it appears.

Since data points for only one size ratio *RD* and one concentration ratio *RC* are available, the general validity of the present theoretical work awaits further experimental study. The results of these experiments will provide data that cover the full range of the size and concentration distributions.

CONCLUSION

This report derives the stresses generated in a highly concentrated shear flow of spherical solids with two sizes. The stresses are considered to be caused predominantly by binary collisions between adjacent particles. The interstitial fluid effect is neglected. The fluid effect could

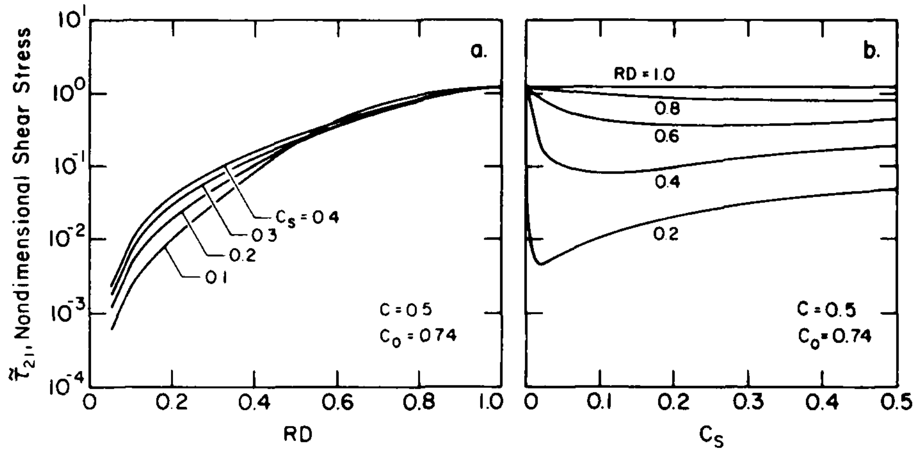

Figure 5. Particle size effect on shear stress.

$$\tau_{22} = -\tau_{21} \frac{(1+\epsilon)^{1/2}}{(0.053 + 0.081\,\mu)^{1/2} \left[\frac{1-\epsilon^2}{4} + \frac{\mu(1+\epsilon)}{\pi} - \frac{\mu^2(1+\epsilon)^2}{4}\right]^{1/2}} \frac{1}{\pi^2 F^{1/2}}$$

$$\cdot \left[2\sqrt{2}\,(RD^{-1} + RC^2 RD^{-13/2}) + 4\,\frac{RC(RD)^{-4} + RC(RD)^{-3}}{(1+RD^3)^{1/2}}\right]. \tag{40}$$

As shown in Figure 5, the nondimensional shear stress

$$\tilde{\tau}_{21} = \frac{2\tau_{21} \left[\frac{1-\epsilon^2}{4} + \frac{\mu(1+\epsilon)}{\pi} - \frac{\mu^2(1+\epsilon)^2}{4}\right]^{1/2} (C_o^{1/3} - C^{1/3})}{C^{4/3}\, \rho_s D_L^2 \left(\frac{du}{dx_2}\right)^2 (1+\epsilon)^{3/2}\, (0.053 + 0.081\,\mu)^{3/2}} \tag{41}$$

is plotted against the two parameters $RD = D_S/D_L$ and C_S for a typical total concentration C.

COMPARISON WITH EXPERIMENTAL DATA

Savage and Sayed (1984) experimentally studied the stresses generated in a couette device filled with dry plastic beads. The results of two tests are shown in Figure 6. Test 1 used nearly uniform beads of 1.32-mm mean diameter. Test 2 used beads of a 7:3 mixture of mean diameters 1.65 mm and 0.6 mm, with a weighted mean of the two sizes of 1.34 mm. The size distributions of these two tests are shown in Figure 7. The straight-line fit of the two sets of experimental data shown in Figure 6 demonstrates that

$$\frac{\tau(\text{two sizes})}{\tau(\text{uniform})} \approx \frac{1}{5}. \tag{42}$$

For this given size distribution, the theoretical prediction using eq 39 gives

$$\frac{\tau(\text{two sizes})}{\tau(\text{uniform})} \approx \frac{1}{10}. \tag{43}$$

$$K_1 = \frac{D_L}{D_S}\left[1 + \frac{G_S}{G_L}\left(\frac{D_L}{D_S}\right)^{3/2}\right]^2 \tag{33}$$

$$K_2 = \frac{(1+\epsilon)(0.053 + 0.081\,\mu)}{\frac{1-\epsilon^2}{4} + \frac{\mu(1+\epsilon)}{\pi} - \frac{\mu^2(1+\epsilon)^2}{4}} \tag{34}$$

and

$$F = \frac{D_L}{D_S} + \frac{s}{D_S} + \left(1 + \frac{s}{D_S}\right)\left(\frac{G_S}{G_L}\right)^2 \frac{D_S}{D_L} + \frac{G_S}{G_L}\left(\frac{D_L}{D_S}\right)^{1/2}\left(1 + \frac{D_L}{D_S}\right)\left(1 + \frac{D_L}{D_S} + \frac{2s}{D_S}\right) / \left[1 + \left(\frac{D_L}{D_S}\right)^3\right] \tag{35}$$

with s given by eq 21.

If we use the results obtained for ν'_L and ν'_S and substitute eq 18, 21, 26, 28, 29 and 32 into eq 2 and 3, the constitutive equations for a granular mixture of spheres with two different diameters are

$$\tau_{21} = C\rho_s D_L^2 \left(\frac{du}{dx_2}\right)^2 \frac{(1+\epsilon)^{3/2}(0.053 + 0.081\,\mu)^{3/2}}{\left[\frac{1-\epsilon^2}{4} + \frac{\mu(1+\epsilon)}{\pi} - \frac{\mu^2(1+\epsilon)^2}{4}\right]^{1/2}} \cdot \frac{(1+RG)RD^{3/2}\,C^{1/3}}{2(1+RG\cdot RD)\,(C_o^{1/3} - C^{1/3})}$$

$$\cdot \frac{F^{3/2}}{[1+(RG)\,RD^{-3/2}]^2} \cdot \frac{1}{1+(RG)RD^3} \tag{36}$$

$$\tau_{22} = -\tau_{21} \frac{(1+\epsilon)^{1/2}}{(0.053 + 0.081\,\mu)^{1/2}\left(\frac{1-\epsilon^2}{4} + \frac{\mu(1+\epsilon)}{\pi} - \frac{\mu^2(1+\epsilon)^2}{4}\right)^{1/2}} \frac{1}{\pi^2 F^{1/2}}$$

$$\cdot \left[2\sqrt{2}(RD^{-1} + RG^2 RD^{-1/2}) + 4\,RG\,\frac{RD^{-1}+1}{(1+RD^3)^{1/2}}\right] \tag{37}$$

in which $RG = G_S/G_L$ and $RD = D_S/D_L$; F is defined in eq 35. Since

$$RG = \frac{G_S}{G_L} = \frac{N_S}{N_L} = \frac{C_S}{C_L}\left(\frac{D_L}{D_S}\right)^3 = RC(RD)^{-3} \tag{38}$$

in which $RC = C_S/C_L$, eq 36 and 37 can be rewritten, respectively, in the following more convenient forms:

$$\tau_{21} = C\rho_s D_L^2 \left(\frac{du}{dx_2}\right)^2 \frac{(1+\epsilon)^{3/2}(0.053 + 0.081\,\mu)^{3/2}}{\left[\frac{1-\epsilon^2}{4} + \frac{\mu(1+\epsilon)}{\pi} - \frac{\mu^2(1+\epsilon)^2}{4}\right]^{1/2}} \cdot \frac{[1+(RC)RD^{-3}]\,C^{1/3}}{2[1+(RC)RD^{-2}]\,(C_o^{1/3} - C^{1/3})}$$

$$\cdot \frac{F^{3/2}}{[1+(RC)\,RD^{-9/2}]^2} \cdot \frac{1}{1+RC} \tag{39}$$

$$\Delta\overline{M}_{LL2} = -\rho_s \frac{\pi D_L^3}{6} \frac{2(1+\epsilon)}{\pi^2} \sqrt{2}\, v'_L$$

$$\Delta\overline{M}_{SS1} = \rho_s \frac{\pi D_S^3}{6} (1+\epsilon)(0.053 + 0.081\,\mu) \frac{du}{dx_2} (D_S + s)$$

$$\Delta\overline{M}_{SS2} = -\rho_s \frac{\pi D_S^3}{6} \frac{2(1+\epsilon)}{\pi^2} \sqrt{2}\, v'_S \,. \tag{28}$$

The momentum transfers between particles of different sizes are obtained similarly. We can derive $\Delta\overline{M}_{LS}$ by considering a stationary large particle being hit by a small particle and statistically averaging the post-collision momentum of the large particle. With the details shown in Appendix A, the results for $\Delta\overline{M}_{LS1,2}$ and $\Delta\overline{M}_{SL1,2}$ are

$$\Delta\overline{M}_{SL1} = \Delta\overline{M}_{LS1} = \rho_s \frac{\pi D_L^3}{6} (1+\epsilon)(0.053 + 0.081\,\mu) \frac{du}{dx_2} \left(\frac{D_S + D_L}{2} + s\right) \frac{2D_S^3}{D_S^3 + D_L^3} \tag{29a}$$

$$\Delta\overline{M}_{SL2} = \Delta\overline{M}_{LS2} = -\rho_s \frac{\pi D_L^3}{6} \frac{4}{\pi^2} \frac{(1+\epsilon) D_S^3}{D_S^3 + D_L^3} \sqrt{{v'_S}^2 + {v'_L}^2} \,. \tag{29b}$$

Average energy dissipation in a pair of spheres–$\overline{E}_{LL}$, $\overline{E}_{LS}$, $\overline{E}_{SS}$

The average energy loss in a pair of colliding particles with identical size, as derived by Shen and Ackermann (1982), becomes

$$\overline{E}_{LL \text{ or } SS} = \rho_s \frac{\pi D_L^3}{6} v'^2_{L \text{ or } S} \left[\frac{1-\epsilon^2}{4} + \frac{\mu(1+\epsilon)}{\pi} - \frac{\mu^2(1+\epsilon)^2}{4}\right] . \tag{30}$$

The average energy loss in a pair of colliding particles with different sizes can be obtained similarly. With the details given in Appendix B, the result is

$$\overline{E}_{LS} = \rho_s \frac{\pi D_L^3}{6} \frac{{v'_L}^2 + {v'_S}^2}{1 + \left(\frac{D_L}{D_S}\right)^3} \left[\frac{1-\epsilon^2}{4} + \frac{\mu(1+\epsilon)}{\pi} - \frac{\mu^2(1+\epsilon)^2}{4}\right] . \tag{31}$$

CONSTITUTIVE EQUATIONS

The functional forms that appear in eq 7-11 are quantified explicitly in eq 18 and 27-31. When we substitute eq 18 and 27-31 into eq 6, together with the equation for the equipartition of energy, eq 12, v'_L and v'_S are

$${v'_L}^2 = \frac{D_L^2 \left(\frac{du}{dx_2}\right)^2}{K_1} K_2 F$$

$${v'_S}^2 = {v'_L}^2 \left(\frac{D_L}{D_S}\right)^3 \tag{32}$$

in which

where N_L is the number of large particles in a unit volume and N_S is that of small particles. The total number of collisions between identical particles is

$$N_{LL} = \frac{N_L N^s_{LL}}{2} \tag{25a}$$

$$N_{SS} = \frac{N_S N^s_{SS}}{2} \tag{25b}$$

since all collisions between like particles are counted twice. Also, the number of total collisions in the flow field is

$$N_{LL} + N_{LS} + N_{SS} = \frac{N_L \frac{\nu'_L}{s} + N_S \frac{\nu'_S}{s}}{2} . \tag{26}$$

If we solve for the seven unknowns—N_{LL}, N_{LS}, N_{SS}, N^s_{LL}, N^s_{LS}, N^s_{SL} and N^s_{SS}—among the seven equations (eq 22-26), the following results are obtained:

$$N_{LL} = N_L \frac{G_L \nu'_L}{GVT} \frac{\nu'_L}{2s} \tag{27a}$$

in which $GVT = G_L \nu'_L + G_S \nu'_S$,

$$N_{LS} = N_S \frac{G_L \nu'_L}{GVT} \frac{\nu'_S}{s} \tag{27b}$$

$$N_{SS} = N_S \frac{G_S \nu'_S}{GVT} \frac{\nu'_S}{2s} \tag{27c}$$

$$N^s_{LL} = \frac{G_L \nu'_L}{GVT} \frac{\nu'_L}{s} \tag{27d}$$

$$N^s_{LS} = \frac{G_S \nu'_S}{GVT} \frac{\nu'_L}{s} \tag{27e}$$

$$N^s_{SL} = \frac{G_L \nu'_L}{GVT} \frac{\nu'_S}{s} \tag{27f}$$

$$N^s_{SS} = \frac{G_S \nu'_S}{GVT} \frac{\nu'_S}{s} . \tag{27g}$$

Average momentum transferred between pairs of spheres—
$\Delta\overline{M}_{LL}$, $\Delta\overline{M}_{LS}$, $\Delta\overline{M}_{SL}$, $\Delta\overline{M}_{SS}$

When two particles of like size collide, the average momentum transfer, as determined by Shen and Ackermann (1982), can be described as

$$\Delta\overline{M}_{LL1} = \rho_s \frac{\pi D_L^3}{6} (1 + \epsilon)(0.053 + 0.081\,\mu) \frac{du}{dx_2} (D_L + s)$$

namely, a particle plus its immediate neighbors, is $G_L[(D_L + D_S)/2] + G_S(D_S)$ for a cell centered around a small particle. For a cell centered around a large particle, the radius is $G_L D_L + G_S[(D_L + D_S)/2]$. In a looser state, as the volume concentration becomes C, the immediate neighboring particles are all moved a distance s away from the center particle. Hence, the corresponding radii for these "cells" become $G_L[((D_L + D_S)/2) + s] + G_S(D_S + s)$ and $G_L(D_L + s) + G_S[((D_L + D_S)/2) + s]$. The mean free path s can therefore be related to the relative concentrations:

$$\frac{C}{C_o} = \frac{(G_L R_L + G_S R_S)^3}{(G_L R_L + G_S R_S + s)^3} \; ; \quad R_L = G_L \quad + G_S \frac{D_L + D_S}{2}$$

$$R_S = G_L \frac{D_L + D_S}{2} + G_S D_S \; . \tag{20}$$

Hence

$$s = \bar{D}\left[\left(\frac{C_o}{C}\right)^{1/3} - 1\right] \text{ with } \bar{D} = G_L R_L + G_S R_S . \tag{21}$$

Frequency of collisions between spheres–N_{LL}, N_{LS}, N_{SS}, N^s_{LL}, N^s_{LS}, N^s_{SL}, N^s_{SS}

Consider a large particle moving in a granular flow as shown in Figure 4. This large particle is being struck by the neighboring small and large particles. N^s_{LS} is the total number of collisions per unit time this large particle experiences from the surrounding small particles and N^s_{LL} is the total number of collisions per unit time this large particle experiences from the surrounding large particles. The collision frequency is proportional to both the number density of the surrounding particles and their fluctuating speed; hence

$$N^s_{LS} = N^s_{LL} \frac{G_S}{G_L} \frac{v'_S}{v'_L} \tag{22}$$

where G_S and G_L are the number percentages of small and large particles in the flow field. Similarly, for a fixed small particle in the flow,

$$N^s_{SS} = N^s_{SL} \frac{G_S}{G_L} \frac{v'_S}{v'_L} \, . \tag{23}$$

The total number of collisions per unit time between large and small particles in a unit time is

$$N_{LS} = N_L \, N^s_{LS} \tag{24a}$$

or

$$N_{LS} = N_S N^s_{SL} \tag{24b}$$

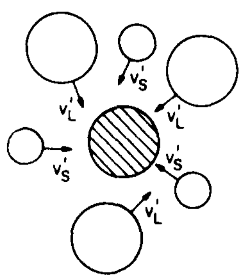

Figure 4. A large sphere being struck by neighboring spheres.

$$K\frac{\pi D^2}{4} = \frac{1}{(\frac{D}{2})}\int_0^{D/2} \frac{\pi(D\sin\psi)^2}{4}\,dx_2 \tag{13}$$

where $D = D_L$ or D_S, $x_2 = D/2\cos\psi$ and ψ is defined in Figure 1. When solved for K, eq 13 shows that the average area of a sphere cut by a random unit surface area is $(\pi/6)D^2$, where $D = D_L$ or D_S. The total areas occupied by large and small particles on a unit area are therefore $p_L(\pi/6)D_L^2$ and $p_S(\pi/6)D_S^2$ respectively.

Let N_L be the number of large particles in a unit volume and N_S be the number of small particles. Assume that in a unit volume the number percentage of large particles is G_L and that of the small particles is G_S. It is easily seen that

$$\frac{N_L}{N_L+N_S} = G_L \quad , \quad \frac{N_S}{N_L+N_S} = G_S \,. \tag{14}$$

The volume concentration of solids is

$$C = N_L\frac{\pi D_L^3}{6} + N_S\frac{\pi D_S^3}{6} \,. \tag{15}$$

The number concentration of large and small particles is solved using eq 14 and 15, resulting in

$$N_L = \frac{C\,G_L}{\frac{\pi}{6}(G_L D_L^3 + G_S D_S^3)} \quad , \quad N_S = \frac{C\,G_S}{\frac{\pi}{6}(G_L D_L^3 + G_S D_S^3)} \,. \tag{16}$$

In a completely random situation, the volume concentration is equal to the area concentration of particle cross sections occupying a unit area; hence

$$C_L = N_L\frac{\pi D_L^3}{6} = p_L\frac{\pi D_L^2}{6} \quad , \quad C_S = N_S\frac{\pi D_S^3}{6} = p_S\frac{\pi D_S^2}{6} \tag{17}$$

or

$$p_L = N_L D_L = \frac{C G_L D_L}{\frac{\pi}{6}(G_L D_L^3 + G_S D_S^3)} \quad , \quad p_S = N_S D_S = \frac{C G_S D_S}{\frac{\pi}{6}(G_L D_L^3 + G_S D_S^3)} \,. \tag{18}$$

It is convenient to now derive another useful kinematic variable, the average gap s between adjacent particles as shown in Figure 1. Let C_o be the concentration corresponding to a densest state for some fixed proportion of large and small particles. In this state all adjacent particles are touching. In a unit volume the number percentage of large particles is G_L and that of small ones is G_S. By definition,

$$G_L = \frac{C_L}{\frac{\pi}{6}D_L^3}\Bigg/\left(\frac{C_L}{\frac{\pi}{6}D_L^3} + \frac{C_S}{\frac{\pi}{6}D_S^3}\right) , \quad G_S = \frac{C_S}{\frac{\pi}{6}D_S^3}\Bigg/\left(\frac{C_L}{\frac{\pi}{6}D_L^3} + \frac{C_S}{\frac{\pi}{6}D_S^3}\right) . \tag{19}$$

In the densest state a given particle is touched by a number of large and small particles; the percentage of large ones is G_L and that of the small ones is G_S. Hence, the average radius of a "cell,"

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